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Variation in the Zeros of a Complex Polynomial

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Abstract

In this paper we prove some extensions of the classical results concerning Enestrom-Kakeya theorem and related analytic functions. Besides several consequences, our results considerably improve the bounds by relaxing the hypothesis in some cases.

Keywords: Polynomials, Enestrom - Kakeya theorem, The sharper bounds, Zeros.

1.INTRODUCTION

The following result due to Enestrom and Kakeya [6],page 136 is well known in the theory of distribution of zeros of polynomials.

Theorem A: If $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \text{------} \geq a_1 \geq a_0 > 0 \quad , \quad a_j \in R \tag{1}$$

Then P(z) has all its zeros in $|z| \leq 1$

Joyal et al [5] extended theorem to the polynomials whose coefficient are monotonic but not necessarily non negative and proved the following:

Theorem B: If $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n such that for some $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq a_{n-2} \geq \text{------} \geq a_1 \geq a_0 \quad , \quad \lambda, a_j \in R, \tag{2}$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq (\lambda a_n - a_0 + |a_0|) \div |a_n|. \tag{3}$$

Among other authors besides Joyal et al[5], Dewan & Govil[3] and Aziz & Zarger[1] also extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non negative.

2.The polynomials with complex coefficients:

Govil and Mc Tune[4] extended the results of Aziz and Zarger[1] to the polynomials with complex coefficients given by:

Theorem C: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j = 0,1,2-----n. If for some $\lambda \geq 1$,

$$\lambda \alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \alpha_0 \quad , \quad \lambda, a_j \in R,$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq (\lambda \alpha_n - \alpha_0 + |\alpha_0| + 2\sum_0^n |\beta_j|) \div |a_n| \tag{4}$$

Also Rather and Shakeel[7] on the lines of Govil & Mc Tune[4] obtained the following result:

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Theorem D: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j = 0,1,2-----n. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \alpha_0 \quad , \qquad \lambda, a_j \in R,$$

then all the zeros of P(z) lie in

$$|z + (\lambda - 1)\frac{\alpha_n}{|a_n|}| \leq (\lambda\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_0^n |\beta_j|) \div |a_n| \qquad (5)$$

Generalizing the above result , Rather & Shakeel[7] also proved the following result:

Theorem E:Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j= 0,1,2-----n. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \alpha_0 \quad ,$$

$$\lambda\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \text{------} \geq \beta_1 \geq \beta_0$$

Then all the zeroes of P(z) lie in

$$|z + \lambda - 1| \leq [\lambda(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|] \div |a_n| \qquad (6)$$

In this paper we consider the generalization of the above theorems and discuss certain properties given by the following:

Theorem 1:Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex co-efficents such that $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for j= 0,1,2-----n. and If for some $\lambda, \mu \geq 1$, $0 < \gamma, \delta \leq 1$

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \text{------} \geq \alpha_1 \geq \alpha_0$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \text{------} \geq \beta_1 \geq \delta\beta_0 \qquad (7)$$

then all the zeros of P(z) lie in the disc:

$$|z + \frac{(\lambda-1)\alpha_n}{a_n}| \leq \frac{1}{|a_n|}[\{(\lambda\alpha_n - \alpha_{n-1})^2 + (\beta_{n-1})^2\}^{1/2} + (\alpha_{n-1} + \beta_{n-1}) + |\alpha_0| + \delta|\beta_0|(1 + \cos\alpha + \sin\alpha) + (\sin\alpha - \cos\alpha)|\beta_0|] \qquad (8)$$

Proof: Consider the polynomial

$$F(z) = (1-z)P(z) = (1-z)(a_0 + a_1z + a_2z^2 + a_3z^3 + \ldots + a_{n-1}z^{n-1} + a_nz^n)$$

$$= (a_0 + a_1z + a_2z^2 + a_3z^3 + \ldots + a_{n-1}z^{n-1} + a_nz^n - a_0z - a_1z^2 - \ldots - a_2z^3 - \ldots - a_{n-1}z^n - a_nz^{n+1})$$

$$= -a_nz^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots\ldots\ldots + (a_2 - a_1)z + a_0$$

$$= -a_nz^{n+1} + (\alpha_n - \alpha_{n-1})z^n + i(\beta_n - \beta_{n-1})z^n + \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1})z^j + i\sum_{j=1}^{j=n-1}(\beta_j - \beta_{j-1})z^j + (\alpha_1 - \alpha_0)z + i(\beta_1 - \beta_0)z + a_0$$

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$= -a_n z^{n+1} - (\lambda\alpha_n - \alpha_n) z^n + (\lambda\alpha_n - \alpha_{n-1}) z^n + i(\beta_n - \beta_{n-1}) z^n + \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1}) z^j + i\sum_{j=1}^{j=n-1}(\beta_j - \beta_{j-1}) z^j + (\alpha_1 - \alpha_0)z + i(\beta_1 - \delta\beta_0)z + i(\delta\beta_0 - \beta_0)z + a_0$

Let $|z|>1$. Then

$|F(z)| \geq |-a_n z^{n+1} - (\lambda\alpha_n - \alpha_n) z^n + (\lambda\alpha_n - \alpha_{n-1}) z^n + i(\beta_n - \beta_{n-1}) z^n + \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1}) z^j + i\sum_{j=1}^{j=n-1}(\beta_j - \beta_{j-1}) z^j + (\alpha_1 - \alpha_0)z + i(\beta_1 - \delta\beta_0)z + i(\delta\beta_0 - \beta_0)z + a_0|$

$\geq |-z^n [\{a_n z + (\lambda-1)\alpha_n\} - \{(\lambda\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})\} - \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1}) z^{j-n}$
$- i\sum_{j=1}^{j=n-1}(\beta_j - \beta_{j-1}) z^{j-n} - (\alpha_1 - \alpha_0) z^{1-n} - i(\beta_1 - \delta\beta_0) z^{1-n} - i(1-\delta)\beta_0 z^{1-n} - a_0 z^{-n}]|$

$= |z|^n |[F_1(\lambda,\alpha,\beta,z) - \{F_2(\lambda,\alpha,\beta) + F_3(\alpha,z) + F_4(\beta,z)\}]|$,
$\geq |z|^n [|F_1(\lambda,\alpha,\beta,z)| - |F_2(\lambda,\alpha,\beta) + F_3(\alpha,z) + F_4(\beta,z)|]$
$\geq [|F_1(\lambda,\alpha,\beta,z)| - |F_2(\lambda,\alpha,\beta) + F_3(\alpha,z) + F_4(\beta,z)|]$
$\leq [|F_1(\lambda,\alpha,\beta,z)| - \{|F_2(\lambda,\alpha,\beta)| + |F_3(\alpha,z)| + |F_4(\beta,z)|\}]$ (by Triangle Inequality)

Therefore, $|F(z)| \leq [|F_1(\lambda,\alpha,\beta,z)| - \{|F_2(\lambda,\alpha,\beta)| + |F_3(\alpha,z)| + |F_4(\beta,z)|\}$ (9)

where ,
$F_1(\lambda,\alpha,\beta,z) = [a_n z + (\lambda-1)\alpha_n]$,
$F_2(\lambda,\alpha,\beta) = (\lambda\alpha_n - \alpha_{n-1}) + i(-\beta_{n-1})$,
$F_3(\alpha,z) = \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1}) z^{j-n} + \alpha_0 z^{-n}$ and
$F_4(\beta,z) = \sum_{j=1}^{j=n-1}(\beta_j - \beta_{j-1}) z^{j-n} + \beta_0 z^{-n}$

Hence $|F_1(\lambda,\alpha,\beta,z)| = |a_n z + (\lambda-1)\alpha_n|$ (10)

Now, the lemma due to Govil & Rehman[5] is given as:
Lemma: If $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some $t>0$, $|ta_j| \geq |a_{j-1}|$, then
$|ta_j - a_{j-1}| \leq \{(|ta_j| - |a_{j-1}|)\cos\alpha + (|ta_j| + |a_{j-1}|)\sin\alpha\}$ (11)

Now, $F_2(\lambda,\alpha,\beta) = (\lambda\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})$
Hence $|F_2(\lambda,\alpha,\beta)| = |(\lambda\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})|$
$= \{(\lambda\alpha_n - \alpha_{n-1})^2 + (-\beta_{n-1})^2\}^{1/2}$ (12)

And $F_3(\alpha,z) = \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1}) z^{j-n} + \alpha_0 z^{-n}$
$= \sum_{j=1}^{j=n-1}(\alpha_j - \alpha_{j-1}) z^{j-n} + (\alpha_0) z^{-n}$
Hence $|F_3(\alpha,z)| \leq \alpha_{n-1} + |(\alpha_0) z^{-n}|$.
$\leq \alpha_{n-1} + |\alpha_0|$ $[since\ |z^{-n}| = \frac{1}{|z^n|} < 1]$ (13)

Similarly $|F_4(\alpha,z)| \leq |\beta_{n-1}| + \delta|\beta_0| + [((\delta\beta_0) - |\beta_0|)cos\alpha + ((\delta\beta_0) + |\beta_0|)sin\alpha]$ (14)

Therefore, from eq.(9), taking into the account of the result of the equations (10),(12),(13) and (14), we have

$|F(z)| \leq |\ a_n z+(\lambda-1)\alpha_n\ | +[\{ (\lambda\alpha_n - \alpha_{n-1})^2 + (\beta n - \beta n - 1)^2\}^{1/2} + \alpha_{n-1} + |\alpha_0| + |\beta_{n-1}| + \delta|\beta_0| + [((\delta\beta_0) - |\beta_0|)cos\alpha + ((\delta\beta_0) + |\beta_0|)sin\alpha$ (15)

Thus for $|z|>1$, $|F(z)|>0$ only if

$|a_n z+(\lambda-1)\alpha_n\ | > [\{ (\lambda\alpha_n - \alpha_{n-1})^2 + (\beta n - \beta n - 1)^2\}^{1/2} + \alpha_{n-1} + |\alpha_0| + |\beta_{n-1}| + \delta|\beta_0| + [((\delta\beta_0) - |\beta_0|)cos\alpha + ((\delta\beta_0) + |\beta_0|)sin\alpha$ (16)

which gives

$$|z+\frac{(\lambda-1)\alpha_n}{a_n}\ | > \frac{1}{|a_n|}[|(\{ (\lambda\alpha_n - \alpha_{n-1}) + i(\beta n - \beta n - 1)| + \alpha_{n-1} + |\alpha_0| + \beta_{n-1} + \delta(1 + cos\alpha + \sin\alpha)|\beta_0| + (\sin\alpha - cos\alpha)|\beta_0|\ ] \tag{17}$$

Above equation shows that the zeros of F(z) having modulii greater than 1 lie in the circle

$$|z+\frac{(\lambda-1)\alpha_n}{a_n}\ | \leq \frac{1}{|a_n|}[\{ (\lambda\alpha_n - \alpha_{n-1})^2 + (\beta n - \beta_{n-1})^2\}^{1/2} + (\alpha_{n-1} + \beta_{n-1}) + |\alpha_0| + \delta\ |\beta_0|(1 + cos\alpha + \sin\alpha) + (\sin\alpha - cos\alpha)|\beta_0|] \tag{18}$$

It can also be verified that the zeros of F(z) whose modulus is less than or equal to one also lie in the circle defined by equation(8) and therefore all the zeros of P(z) lying in the disc given by equation(8).Now, when $\alpha = 0$, then L.H.S. becomes

$$|z+\frac{(\lambda-1)\alpha_n}{a_n}\ | \leq \frac{1}{|a_n|}[\{ (\lambda\alpha_n - \alpha_{n-1})^2 + (\beta_{n-1})^2\}^{1/2} + (\alpha_{n-1} + \beta_{n-1}) + |\alpha_0| + \delta\ |\beta_0|(1 + cos\alpha + \sin\alpha) + (\sin\alpha - cos\alpha)|\beta_0|]$$

which proves Th. 1.

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